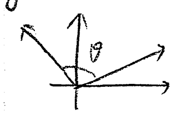
Lesson 19

§5.1 §5.3

$\vec{v}_2 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$



$v_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Dot product (Standard Inner product)

$= v_1 \cdot v_2 = v_1^T v_2 = x_1 y_1 + x_2 y_2$.

<u>length</u> or <u>magnitude</u> of $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$\|v\| = (v \cdot v)^{\frac{1}{2}} = \sqrt{x^2 + y^2}$

<u>Distance</u> between v_1 and v_2

$d(u,v) = \| v_1 - v_2 \| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

<u>Angle</u> between v_1 and v_2

$v_1 \cdot v_2 = \|v_1\| \|v_2\| \cos\theta$

$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\| \|v_2\|}$ $\quad \theta \in [0, \pi]$ $\qquad$ <u>why not</u> $[0, 2\pi]$?

$\cos\theta \in [-1, 1]$ $\quad \begin{cases} \cos\theta \geq 0 & \theta \in [0, \frac{\pi}{2}] \\ \cos\theta \leq 0 & \theta \in [\frac{\pi}{2}, \pi] \end{cases}$

Orthogonal / perpendicular.

$v_1 \cdot v_2 = 0$

For two vector $v_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $v_2 = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$.

$v_1 \cdot v_2 = v_1^T v_2 = x_1 y_1 + \cdots + x_n y_n$

All subjects are defined in a similar way.

EX: Find all vectors orthogonal to $\underset{v_1}{\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}}$ $\underset{v_2}{\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}}$ $\underset{v_3}{\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}}$

Sol: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = v \in \mathbb{R}^3$

$v \cdot v_i = 0 \Rightarrow A = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 1 & -3 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$A \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Compare with linear combination

Def: u is called a unit vector of $\mathbb{R}^n$ if $\|u\| = 1$

Ex: $u \neq 0$ $\frac{u}{\|u\|}$ is a unit vector.

Properties of standard inner product.

(a) $u \cdot u \geq 0$ $\quad u \cdot u = 0$ iff $u = 0$

(b) $u \cdot v = v \cdot u$

(c) $(u+v) \cdot w = u \cdot w + v \cdot w$

(d) $(cu) \cdot v = c(u \cdot v)$ $\quad c \in \mathbb{R}$

For any V.S. V, an inner product (I.P) is a map: $(u, v) \to \mathbb{R}$ satisfying (a) – (d)

EX: $(u, v) := u \cdot v$ the dot product is a inner product.

Remark: (i) $(u, v+w) = (v+w, u) = (v, u) + (w, u) = (u, v) + (u, w)$

(ii) (d) $\Rightarrow$ $(u, cv) = c(u \cdot v) \boxed{= (cu, v)}$

Examples in $\mathbb{R}^n$

EX: $\{v_1 \cdots v_n\}$ a basis for $\mathbb{R}^n$

$u = \sum a_i v_i$ $\quad w = \sum b_i v_i$ $\quad (u, w) := \sum a_i b_i$ is a I.P.

Proof. (b) – (d) is clearly $\quad u \cdot u = \sum a_i^2 \geq 0$

$u \cdot u = 0 \Rightarrow a_i^2 = 0 \Rightarrow a_i = 0 \Rightarrow u = \sum 0 \cdot v_i = 0$.

How to interpret a non-standard inner product?

standard I.P $\to$ a measure for length. $\to$ different in north pole / equator.

$(\cdot, \cdot)$ I.P $\quad S = \{v_1 \cdots v_n\}$ a basis $\quad u = \sum a_i v_i \quad w = \sum b_i v_i$

$(u, w) = [u]_S^T \, C \, [w]_S$ $\quad \leftarrow$ Theorem 5.2

$[u]_S$ is the coordinates w.r.t. S

$C = [c_{ij}]$ $\quad c_{ij} = (v_i, v_j)$

$\left.\begin{array}{l} \text{(b)} \Rightarrow C \text{ sym} \\ \text{(a)} \Rightarrow x^T C x > 0 \quad x \neq 0 \end{array}\right\}$ C positive definite

Def: C is called the matrix representation of $(\cdot, \cdot)$ w.r.t S.

Theorem: $(\cdot , \cdot)$ is a I.P in $\mathbb{R}^n$ iff $(u, v) = u^T C v$
C positive definite.

→ Inner product in Non-Euclidean spaces

EX: $(f, g) \mapsto \int_a^b f(t)\, g(t)\, dt$ is a I.P on
$P_2(I) :=$ set of all poly of deg ≤ 2 on $I := [a, b]$

Exer Check $(\cdot , \cdot)$ is also I.P on $C^k(I)$, $P(I)$, $P_n(I)$

Q What is the matrix representation of $(\cdot , \cdot)$ on $P_2(I)$
A: $at^2 + bt + c \mapsto \begin{bmatrix} a \\ b \end{bmatrix}$ $\{t^2, t, 1\}$ standard basis

3×3 sym. matrices $\Downarrow$ 6 entries

$$\begin{cases} (t^2, t^2) = \int_0^1 t^4\,dt = \frac{1}{5} & (t, t) = \int_0^1 t^2\,dt = \frac{1}{3} \\ (t^2, t) = \int_0^1 t^3\,dt = \frac{1}{4} & (t, 1) = \int_0^1 t\,dt = \frac{1}{2} \\ (t^2, 1) = \int_0^1 t^2\,dt = \frac{1}{3} & (1, 1) = \int_0^1 1\,dt = 1 \end{cases}$$

$f = a_2 t^2 + a_1 t + a_0 \qquad g = b_2 t^2 + b_1 t + b_0$

$$(f, g) = [a_2 \;\; a_1 \;\; a_0] \begin{array}{c} \left[\begin{array}{ccc} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{array}\right] \begin{array}{l} \leftarrow v_1 \\ \leftarrow v_2 \\ \leftarrow v_3 \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 \end{array} \end{array} \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

Def: A V.S. V with an IP is called an inner product space. If $\dim V < \infty$, V is called an Euclidean space.

length: $\|u\| := \sqrt{(u, u)}$

Distance: $d(u, v) = \|u - v\|$

Angle: $\cos\theta = \frac{(u, v)}{\|u\|\,\|v\|} \qquad \cos\theta \in [-1, 1]$

Orthogonal: $(u, v) = 0 \qquad$ Unit vector: $u = \frac{x}{\|x\|}$

Def: $S = \{v_1 \cdots v_n\} \subseteq V$ is called orthogonal if

① $(v_i, v_j) = 0 \quad i \neq j$

S is called orthonormal if in addition

② $\|v_i\| = 1$

EX1: $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal.

$\tilde{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix} \quad \tilde{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \quad v_3$ is orthonormal.

Thm: $S = \{v_1 - v_n\}$ orthogonal $v_i \neq 0 \Rightarrow S$ L.I. (Give proof)

Cor S orthonormal $\Rightarrow S$ L.I. Remark: NOT True for orthogonal.

EX1 (Continuous) $\{v_1\ v_2\ v_3\}$ orthogonal basis $\{\tilde{v}_1 . \tilde{v}_2\ v_3\}$ orthonormal basis.

Properties: (1) $|(u, v)| \leq \|u\|\,\|v\|$ (Cauchy-Schwartz Inequality)

(2) $\|u + v\| \leq \|u\| + \|v\|$ (Triangle Inequality)

Example 14 Post